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Nonstandard (Non-σ-Additive) Probabilities in Algebraic Quantum Field Theory

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Traditionally, physicists deduce the observational (physical) meaning of probabilistic predictions from the implicit assumption that the *well-defined* events whose probabilities are 0 never occur. For example, the conclusion that in a potentially infinite sequence of identical experiments with probability 0.5 (like coin tossing) the frequency of heads tends to 0.5 follows from the theorem that sequences for which the frequencies do not tend to 0.5 occur with probability 0. Similarly, the conclusion that in quantum mechanics, measuring a quantity always results in a number from its spectrum is justified by the fact that the probability of getting a number outside the spectrum is 0. In the mid-60s, a consistent formalization of this assumption was proposed by Kolmogorov and Martin-Löf, who defined a *random* element of a probability space as an element that does not belong to any definable set of probability 0 (definable in some reasonable sense). This formalization is based on the fact that traditional probability measures are σ-additive, i.e., that the union of countably many sets of probability 0 has measure 0. In quantum mechanics with infinitely many degrees of freedom (e.g., in quantum field theory) and in statistical physics one must often consider non-σ-additive measures, for which the Martin-Löf's definition does not apply. Many such measures can be defined as "limits" of standard probability distributions. In this paper, we formalize the notion of a random element for such finitely-additive probability measures, and thus explain the observational (physical) meaning of such probabilities.

1. OBSERVATIONAL (PHYSICAL) MEANING OF TRADITIONAL (σ-ADDITIVE) PROBABILITY MEASURES: RANDOMNESS IN THE SENSE OF KOLMOGOROV–MARTIN-LÖF

1.1. How to Describe When Experiments Are consistent with a Probabilistic Theory: Idea

In classical (prequantum) physics, predictions were usually deterministic. For such predictions, there is no problem of checking whether a theory

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is consistent with the experiment: if the predicted event occurs, the theory is confirmed (so far); if it does not occur, the theory is rejected.

In modern physics, the majority of the predictions are of *probabilistic* nature: instead of predicting what exactly is going to happen, physics can, at best, predict the *probabilities* of different outcomes. In this case, it is not so clear whether an experiment is consistent with the theory or not. For example, if the hypothesis is that we have a fair coin (with probability of heads 0.5), what sequences of actual coin-tossing results are consistent with this hypothesis?

For one, if the coin is really fair, in this sequence the frequency f_n of heads among the first n toss results must tend to 0.5 as $n \to \infty$. Why? Because it can be mathematically proven that for a probability measure that describes a fair coin, the probability that $f_n \not\to 0.5$ is 0.

All other statistical laws are justified in exactly the same manner: *we prove that some well-defined event has a probability* 0, *and from this we conclude that this event simply cannot occur.*

This conclusion is based on an implicit assumption that a well-defined event with 0 probability cannot occur.

A typical quantum example is: the conclusion that in quantum mechanics, measuring a quantity always results in a number from its spectrum is justified by the fact that the probability of getting a number outside the spectrum is 0.

1.2. This Idea May Seem Inconsistent

At first glance, it may seem that this idea will not work for simple continuous probability distributions on a real line, e.g., for the normal distribution or for the uniform distribution on some interval. In these distributions, for every real number x, the probability of getting exactly this real number is 0.

So it may seem that no real number can occur at all, and that, therefore, the above assumption is inconsistent.

1.3. This Idea Is Consistent If We Restrict It to "Well-Defined" Events Only

The situation with simple continuous distributions of the real line (e.g., with the standard Gaussian distribution) is not so bad if we properly interpret the term "well-defined" in our main assumption as meaning that this event can be uniquely defined by a text in some formal language.

For example, the real numbers 0, 1, 0.5, e, etc., are all well defined. For such well-defined numbers, the conclusion that these numbers *cannot* occur as a result of a normally distributed random process makes perfect physical sense. To be more precise, if we measured the unknown physical

quantity with an accuracy 1 and we got the result $\tilde{x} = 1$, then it could still happen that the *actual* value of the measured quantity is random relative to the standard Gaussian distribution. However, if we measure the same quantity x with better and better accuracy and after each measurement we still get 1 (e.g., if we get 1, 1.0, 1.00, etc.), then after a few measurements, every physicist would conclude that the actual value of this quantity x is *not* random (because, according to physicists' intuition, 1.000 ... is not a random number).

Here, 1 was simply an example. If we get 0, or 0.5, or e, a physicist will also conclude that this is *not* a random number.

If we require that *well-defined* events with probability 0 do not occur, then we *no* longer have a *contradiction*: Indeed, there are only countably many texts in any given language, and therefore there exist no more than countably many events of probability 0 (in particular, there are only countably many definable real numbers). If we exclude all these events, then totally, we are excluding the countable union of events of probability 0; the remainder is not empty (it actually has probability 1).

Let us describe how this idea is formalized.

1.4. Random Element: Formalization of the Observational Meaning of Probabilistic Theories

To describe which probabilistic theories are consistent with experiments and which are not, we must formalize the implicit assumption that an event with 0 probability cannot occur.

In mid-60s, a consistent formalization of this assumption was proposed by Kolmogorov and Martin-Löf, who defined a *random* element of the probability space as an element that does not belong to any definable set of probability 0 (definable in some reasonable sense).

The formal definition of "random" was proposed by Kolmogorov's student Martin-Löf (1966); for current coverage of this topic, see, e.g., Li and Vitányi (1997). [We will be using a version of this definition proposed by Benioff (1976); see also Kreinovich and Longpré (1996)].

Definition 1.1. Let a mathematical language L be fixed (e.g., language of set theory, or language of recursive objects).

- Sets defined by formulas from L (i.e., sets of the type $\{x|P(x)\}$, where $P(x)$ is a formula from L) will be called (L-)*definable* (or simply *definable*, for short).
- Let μ be a probability measure on a set X. An element $x \in X$ is called *random* w.r.t. μ if x does not belong to any definable set of μ-measure 0.

Comment. For recursive L, we get the original Kolmogorov–Martin-Löf definition of randomness.

1.5. σ-Additivity Is Important for this Definition

The crucial point is to prove that random elements exist, and that, moreover, the set of all random elements has probability 1 (i.e., that almost all elements are random). To prove that, Martin-Löf used the fact that traditional probability measures are σ-*additive,* i.e., that the union of countably many sets of probability 0 has measure 0.

Indeed, in every language, there are at most countably many words. Since different definable sets are described by different formulas, there are at most countably many different definable sets of measure 0. Hence, the complement to the set of all random elements is the union of countably many sets of measure 0 and thus itself a set of measure 0.

2. FINITELY-ADDITIVE MEASURES NATURALLY EMERGE IN QUANTUM FIELD THEORY AND IN STATISTICAL PHYSICS

Kolmogorov's probability theory and Kolmogorov–Martin-Löf definition of a random sequence are based on the assumption that the probability measure μ is σ-additive, i.e., that the union of countably many sets of probability 0 also has probability 0. Another (equivalent) way of describing σ-additivity is that if we have a countable sequence $A_1, \ldots A_n, \ldots$ of measurable pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$\mu(A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots) = \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n) + \ldots$$

In modern physics, however, probabilities are often only *finitely* additive, i.e.,

$$\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n)$$

is true only for *finite* families of pairwise disjoint sets. We will show that for such probability measures, Martin-Löf's definition does not work. Before we describe why, let us briefly enumerate the cases when such finitely-additive measures emerge (Wightman, 1976).

2.1. Finitely-Additive Measures in the Description of the Classical Limit of Quantum Systems

2.1.1. Why Classical Limit

It is well known that for the same system, quantum equations are much more complicated to solve than the corresponding classical ones. The main

method of solving the corresponding quantum problems is to take into consideration the fact that many quantum effects can usually be neglected, and we can replace the precise quantum description of the corresponding effects by its *classical limit*.

From the physical viewpoint, quantum effects are caused by the fact that Planck's constant $\hbar$ is different from 0. For example, since $\hbar > 0$, the Heisenberg inequality $\Delta p \cdot \Delta x \geq \hbar$ causes the uncertainties Δp and Δx in momentum p and coordinate x to be different from 0; thus, in contrast to classical physics, a quantum particle cannot have precise values of both p and x. Hence, ideally, to define a classical limit, we must be able to replace Planck's constant in the equations of quantum physics by 0. From the mathematical viewpoint, since Schrödinger equation (the basic equation of quantum mechanics) includes division by $\hbar$, we cannot simply divide by 0. We must, instead, do the following:

- Rewrite the quantum equations in such a way that all occurrences of the Planck constant have a symbol $\hbar$ instead of its actual numerical value.
- Replace this symbol $\hbar$ by smaller and smaller values, and compute the physical consequences of the resulting equations.
- Take the limit values of these effects (when $\hbar \to 0$) as the desired classical limit.

2.1.2. Pointwise Particle

Let us consider the classical limit of a 1D particle (which is the simplest possible quantum object). In quantum mechanics, a particle is never located with a 100% certainty at a single point. Instead, it can appear at different locations with different probabilities. The corresponding probability density $\rho(x)$ is characterized by a wave function $\psi(x)$ as $\rho(x) = |\psi(x)|^2$. Fourier transform of the wave function gives us the distribution in momentum space.

In the classical limit, we must have classical pointwise particles. Thus, in the classical limit $\hbar \to 0$, the wave function must become concentrated at a single point. The corresponding limit probability density can be described by a delta-function $\delta(x)$ [so that, crudely speaking, $\psi(x) = \sqrt{\delta(x)}$].

As $\hbar \to 0$, the wave function becomes closer and closer to this pointwise limit, and the corresponding probability distribution in the momentum space becomes distributed across larger and larger intervals $[-p, p]$. The wave function for p is, in quantum mechanics, equal to the Fourier transform of the wave function for x. So, when the wave function for x tends to the delta-function, the corresponding wave function for p tends to the Fourier transform of the delta-function, i.e., to a constant. Thus, the limit probability density is constant throughout the entire real line.

If we had a constant density over a certain interval, then we would say that we have a uniform distribution over this interval. Thus, since in the limit we have a constant density over the entire real line, we can say that in this limit we get a "uniform" distribution over the real line. In more precise terms, we can conclude that the probability of the momentum p to be in an arbitrary set A tends to the limit $\mu(A) = \lim \mu_0(A \cap [-n, n]/\mu_0([-n, n])$, where $\mu_0(A)$ is the standard Lebesque measure.

One can easily check that this probability measure $\mu(A)$ is finitely additive. However, thus-defined probabilities $\mu(A)$ are *not* σ-additive: Indeed:

- The sets $A_i = [i, i + 1)$ are pairwise disjoint.
- For each of them, $\mu(A_i) = \lim_n (1/(2n)) = 0$.
- But (since $R = \cup A_i$), for their union $\cup A_i$, we have

$$\mu(\cup A_i) = \mu(R) = 1 \neq 0 = \sum_{i=-\infty}^{\infty} \mu(A_i) = \sum_{i=-\infty}^{\infty} 0$$

In reality, of course, this finitely-additive measure is only an *approximation* to the actual σ-additive probability measure and therefore, in principle, if we will not be able to define randomness in the sense of the "limit" measure μ, we will always be able to use the original probability measure μ_n instead. However, for all practical purposes, we can replace the actual measure μ_n by its finitely additive approximation μ: when n is large enough, all statistical characteristics of μ_n are as close as possible to the characteristics of the limit measure μ.

Since all the observable characteristics of μ are indistinguishable from the characteristics of μ_n, from the physical viewpoint, both μ and μ_n give a good description of the system. It is therefore desirable to be able to define a random element w.r.t. μ in such a way that random elements w.r.t. μ will be approximately the same ones as random elements w.r.t. μ_n.

2.1.3. Wave

A similar argument can be repeated for a *planar wave*. A planar wave is usually described by a periodic wave function $\phi(x) = a_0 \exp(i\mathbf{k} \cdot \mathbf{x})$.

This description is used in standard textbooks on quantum theory (see, e.g., Landau and Lifschitz, 1965) and it is a pretty good description of certain physical phenomena. For this state, the probability density $\rho(x) = |\phi(x)|^2 = |a_0|^2$ has the exact same value for all points x. In this sense, we have a uniform distribution on the real line. We can approximate this distribution by uniform distributions located on an interval $[-n, n]$ for $n \to \infty$. For every n, the condition that the total probability of being somewhere must add up to 1 leads to $|a_0|^2 = 1/2n$, so in the limit, we have the same finitely-additive

probability distribution as the one that describes momentum of a pointwise particle.

In both cases, finitely-additive probability measures correspond to the states that are outside the standard Hilbert space of states (although they can be represented as limits of states from the standard Hilbert space).

2.2. Algebraic (Finitely-Additive) States in Quantum Field Theory

For a single particle, and, moreover, for an arbitrary quantum system with *finitely many degrees of freedom*, finitely-additive measures and corresponding non-Hilbert states are a good approximation, a good computational tool, but strictly speaking, they are not necessary for describing the system. However, when we turn to systems with *infinitely many degrees of freedom* (e.g., to quantum field theory), the situation drastically changes: if we continue to use states from the standard Hilbert space, we end up getting "divergences," meaningless infinite values that indicate that the resulting states are *outside* the standard Hilbert space (Emch, 1972, 1984, Wightman, 1976). Such non-Hilbert states cannot be easily described by the standard von Neumann formalism; they need special algebraic generalizations; because of that, such states are usually called *algebraic states,*.

Non-Hilbert spaces usually correspond to nonstandard, i.e., finitely-additive probability measures. Therefore, if we want to understand what exactly these physical predictions mean in observational terms, we must be able to generalize the notion of a random element to such measures.

Comment. Initially, some mathematicians viewed these "divergences" as indications that the existing formalism is inconsistent and that an alternative approach is necessary. From this viewpoint, our problem may seem misstated: instead of defining random elements w.r.t. finitely-additive measures, why not find an alternative theory that would lead to a σ-additive probability measure?

It turned out, however, that in principle, the old (seemingly ad hoc) formalisms can be reformulated in a mathematically consistent manner; a large part of these formalisms can be reformulated in terms of the algebraic quantum field theory (Emch, 1972; 1984; Wightman, 1976), and practically all parts of these formalisms can be reformulated in terms of *nonstandard analysis* (Robinson and Keleman, 1972; Kreinovich, 1980), a theory proposed by Robinson (1974, see also Davis, 1977) to formalize the physicists' intuition about actual infinities and actual infinitesimals. So, our problem *is* a real problem.

2.3. Algebraic States Are Probably Needed for Finkelstein's Quantum Topology

To some extent, quantum field theory can be viewed as a phenomenological theory, because it is based on the underlying notion of the nonquantized

space-time. From the methodological viewpoint, it is desirable to reformulate all physics in terms of the most fundamental physical notion: the notion of causality. This program has led to many interesting ideas and results (see, e.g., Finkelstein and Gibbs, 1993; Finkelstein, 1996).

Do we need algebraic states for this formalism? For more standard quantum formalisms, the answer would be: yes, if this formalism describes systems with infinitely many degrees of freedom. Finkelstein's description is based on causality, not on real numbers and vector spaces, so we cannot directly tell how many degrees of freedom we describe; however, the fact that with this formalism we can describe fields and other systems that traditionally require infinitely many degrees of freedom to describe makes it reasonable to assume that *for Finkelstein's formalism*, algebraic states (and *finitely-additive measures*) *are* also *necessary.*

2.4. Algebraic States in Statistical Physics

Standard distributions from traditional statistical physics are derived from the (sometimes implicit) assumption that the initial locations are distributed uniformly in R^3 (or, for N particles, in R^{3N}). From the viewpoint of precise probability theory, there is no such distribution, so, in order to formally justify these physical derivations, mathematicians describe probability measures uniformly distributed on a set of volume V and then tend V to ∞ (see, e.g., Emch, 1972, 1984; Wightman, 1976; Rényi, 1970).

From the physical viewpoint, however, this limit transition is nothing more than a mathematical trick whose goal is to formalize the physical notion of uniform distribution on a space. It is therefore desirable to make observational predictions based on this physical distribution rather than on its mathematical approximation. Again, we need to define the notion of a random element for finitely-additive measures.

3. THE PROBLEM: KOLMOGOROV–MARTIN-LÖF RANDOMNESS IS NOT DIRECTLY APPLICABLE TO FINITELY-ADDITIVE MEASURES

In the previous section we showed that in many physical situations, the probability measure μ is only finitely additive, and that it is desirable to describe elements random w.r.t. this measure μ.

At first glance, it may seem that the same informal idea can be applied here: we can call an element random if it does not belong to any definable set of measure 0. We have seen that for σ-additive measures, this definition leads to a reasonable class of "random" elements. However, as we will see in a moment, for finitely-additive measures, a similar definition would be

meaningless (because there would be no elements that are "random" in the sense of this definition).

As an example, let us consider the uniform distribution μ of a real line. This distribution can be defined as a limit of uniform distributions on the intervals $[-n, n]$ as $n \to \infty$. In other words, for every set $A \subseteq R$, we can define $\mu(A)$ as

$$\mu(A) = \lim_{n \to \infty} \frac{\mu_0(A \cap [-n, n])}{\mu_0([-n, n])}$$

where μ_0 is the standard Lebesgue measure on the real line. For this distribution and for every integer m, the probability of the interval $[m, m + 1)$ is the limit of $1/(2n)$, i.e., 0. Thus, if we apply the above definition of a random element, we will have to conclude that a random element x does not belong to any interval $[m, m + 1)$. But these intervals cover the entire real line, so there are simply no real numbers that would be outside all of them. Thus, in the sense of this definition, there are simply no random elements.

A similar construction shows that no random elements exist for a uniform distribution on R^k that is used in statistical physics. So, we have a problem: *How do we define randomness for non-σ-additive probability measures*?

4. OUR SOLUTION: A NEW DEFINITION OF RANDOMNESS FOR "LIMIT" PROBABILITY MEASURES

4.1. Solution: Simplest Case

We have seen that many finitely-additive measures emerging in quantum mechanics with finitely many degrees of freedom (e.g., in quantum field theory) and in statistical physics are "limits" of standard probability distributions. Let us show how to define the notion of a random element for such "limit" measures. Before doing that, let us first formalize what "limit" measure means.

Definition 4.1. We say that a finitely additive measure μ on a set X is a simple limit measure if there exist:

- a definable sequence of sets $V_1 \subseteq V_2 \subseteq \ldots V_n \subseteq \ldots$ such that $\cup V_i = X$;
- a definable sequence of σ-additive probability measures μ_i defined on V_i such that for every $i < j$, a set $A \subseteq V_i$ is measurable w.r.t. μ_i iff it is measurable w.r.t. μ_j; and
- a definable sequence of positive real numbers λ_{ij}, $i < j$, such that for every $A \subseteq V_i$, we have $\mu_i(A) = \lambda_{ij} \cdot \mu_j(A)$;

such that for every set $A \subseteq X$ for which the measure $\mu(A)$ is defined, we have $\mu(A) = \lim_n \mu_n(A \cup V_n)$.

Example. A "uniform measure on the real line" ($X = R$) can be defined as follows: For each n, we take the interval $[-n, n]$ as V_n, and a uniform distribution on V_n as μ_n. Then the equality $\mu_i(A) = \lambda_{ij} \cdot \mu_j(A)$ holds for $\lambda_{ij} = j/i$.

Since the measure μ is defined as a limit of σ-additive measures μ_n, it is natural to define the set R of all elements random w.r.t. μ as a "limit" of the sets R_n random w.r.t. μ_n. It turns out that this sequence of sets R_i is monotonically nondecreasing ($R_i \subseteq R_2 \subseteq \ldots \subseteq R_n \subseteq \ldots$; see Proposition 1 below), and therefore we can define this "limit" set as simply the union $R = \cup_{n=1}^{\infty} R_n$:

Proposition 1. Let μ be a simple limit measure, $i < j$ be integers, and let $x \in V_i$ be random w.r.t. μ_i. Then, x will be also random w.r.t. μ_j.

Comment. For the reader's convenience, all proofs are placed in the special Proofs section at the end of the paper.

Definition 4.2. Let μ be a simple limit measure on the set X. We say that an element x is *random* w.r.t. μ if it is random w.r.t. one of the measures μ_i.

Before we analyze this definition, let us first check that the set of random elements is indeed nonempty (since the previous attempt at defining sequences random w.r.t. a limit measure was ruined by the discovery that no elements are random in that sense). This is easy to check because according to the new definition, the set of random elements is defined as a *union* of sets of elements random in the traditional sense, and it therefore contains each of these (nonempty) sets, so it is nonempty itself.

4.2. Solution: General Case

In the above simple case, we assumed that in the sequence μ_n that defines μ, for every $i < j$, the restrictions of measures μ_i and μ_j on V_i are *linearly* related. It turns out that for our purpose (i.e., for defining a random element), it is quite sufficient to assume a much weaker condition: that the restrictions of μ_i and μ_j on V_i are *absolutely continuous* w.r.t. each other (see e.g., Edwards, 1995, Chapter 4), i.e., a set $A \subseteq V_i$ has μ_i-measure 0 iff it has μ_j-measure 0:

Definition 4.3. We say that a finitely-additive measure μ on a set X is a *limit measure* if there exist:

- a definable sequence of sets $V_1 \subseteq V_2 \subseteq \ldots V_n \subseteq \ldots$ such that $\cup V_i = X$;

- a definable sequence of σ-additive probability measures μ_i defined on V_i such that for every $i < j$, a set $A \subseteq V_i$ is measurable w.r.t. μ_i iff it is measurable w.r.t. μ_j; and a set $A \subseteq V_i$ has μ_i-measure 0 iff it has μ_j-measure 0 (i.e., the restrictions of μ_i and μ_j to V_i are absolutely continuous w.r.t. each other);

such that for every set $A \subseteq X$ for which the measure $\mu(A)$ is defined, we have $\mu(A) = \lim_n \mu_n(A \cup V_n)$.

Similarly to the simple limit case, we can show that the sets R_n of elements random w.r.t. μ_n form a monotonic sequence, and thus we can apply Definition 4,2 to define an element random w.r.t. the limit measure:

Proposition 2. Let μ be a limit measure, $i < j$ be integers, and let $x \in V_i$ be random w.r.t. μ_i. Then x will also be random w.r.t. μ_j.

Definition 4.4. Let μ be a limit measure on the set X. We say that an element x is *random* w.r.t. μ if it is random w.r.t. one of the measures μ_i.

4.3. Solution Reformulated: Idea

From the "*practical*" viewpoint, our definition is acceptable. From the *fundamental*, methodological viewpoint, it has the following (slight) drawback: namely, when defining a random element w.r.t. a finitely additive measure μ, we explicitly used a sequence μ_n whose limit is μ. Intuitively, a measure μ can be defined as a limit in different ways: e.g., the "uniform" distribution on the real line can be defined either as a limit of uniform distributions on the intervals $[-n, n]$ or as a limit of uniform distributions located on the intervals $[-(n + 1/2), (n + 1/2)]$. It is desirable to require that the notion of a random element should not depend on the specific choice of the approximating sequence of measures. We will show that this requirement is indeed satisfied for the above definition.

To show that, we will show that randomness in the sense of this definition is equivalent to randomness in the sense of some σ-additive (but not probabilistic) measure μ_0 that is defined from μ.

Historical comment. The idea of using a σ-additive nonprobabilistic measure μ_0 to describe finitely additive probability measure μ is well known in probability theory: see, e.g., Chapter 2 of the classic monograph by Rényi (1970) or a more modern exposition in Hartigan (1983). We will show that our definition of randomness relative to μ can indeed be reformulated in terms of this auxiliary σ-additive measure μ_0.

4.4. Solution Reformulated: Simple Limit Measures

Definition 4.5. Let μ_0 be a σ-additive measure (not necessarily a probability measure) on a set X. An element $x \in X$ is called *random* w.r.t. μ_0 if x does not belong to any definable set of μ_0-measure 0.

Definition 4.6 (Rényi, 1970; Hartigan, 1983). Let μ be a simple limit measure. We can then define, for every set $A \subseteq V_i$, $\mu_0(A) = \lambda_{1i} \cdot \mu_i(A)$.

Comment. One can easily check that this definition is consistent [i.e., for the sets A that belong to both V_i and V_j, the corresponding formulas lead to the same value $\mu_0(A)$], and that this definition defines an additive measure that can be expanded to a σ-additive measure μ_0 on X. We will call this measure *associated* with μ. For example, for a uniform distribution on the real line, μ_0 is simply a standard (Lebesgue) measure on R (maybe times a constant). Similarly, for the statistical physics case, we get the standard Lebesgue measure on R^k. For these examples, the limit measure does not depend on the (physically meaningful) choice of a limit sequence: e.g., $[-n, n]$ and $[-(n + 1/2), n + 1/2]$ lead to the same Lebesgue measure. Therefore, to show that our definition of randomness does not depend on the limit sequence μ_n, it is sufficient to show that this definition can be reformulated in terms of μ_0 only. This reformulation is provided by the following proposition:

Proposition 3. Let μ be a simple limit measure, and let μ_0 be an associated σ-additive measure. Then an element x is random w.r.t. μ if and only if it is random w.r.t. μ_0.

Comment. From the above text, the reader may get the wrong idea that finitely-additive probabilities are reasonably easy to handle, and that the results of using them are practically always intuitively clear. Alas, this is not always the case:

- It is worth mentioning that two different limit measures $\mu \neq \nu$ can lead to the same associated σ-additive measure. For example, let us take $\mu = \lim \mu_n$ and $\nu = \lim \nu_n$, where μ_n is a uniform distribution on the interval $[-n, n]$, while ν_n is a uniform distribution on the interval $[-n, 2n]$. These two measures lead to the same associated measure: namely, to the Lebesgue measure μ_0. However, these two measures are different: e.g., the probability of an element to be positive is equal to 1/2 according to $\mu = \lim \mu_n$, but to 1/3 according to $\nu = \lim \nu_n$.
- Finitely additive measures also lead to counterintuitive *conditional* probabilities (see, e.g., Schervish *et al.*, 1984).

To handle such problems, alternative (nonlimit) approaches to describing finitely-additive probabilities have been proposed (see, e.g., Heath and Sudderth, 1978).

4.5. Solution Reformulated: General Case

Definition 4.7. Let μ be a limit measure. We can then define, for every set $A \subseteq V_i - V_{i-1}$, $\mu_0(A) = \mu_i(A)$.

Comment. One can easily check that this definition defines an additive measure that can be expanded to a σ-additive measure μ_0 on X. We will call this measure *associated* with μ.

Proposition 4. Let μ be a limit measure, and let μ_0 be an associated σ-additive measure. Then an element x is random w.r.t. μ if and only if it is random w.r.t. μ_0.

5. PROOFS

5.1. Proof of Proposition 1

Let $x \in V_i$ be random w.r.t. μ_i, and let $i < j$. Let us show that x is random w.r.t. μ_j as well, i.e., that x does not belong to any set $A \subseteq V_j$ of μ_j-measure 0. Indeed, since x is an element of V_i, the only possibility for x to belong to A is to belong to $A \cap V_i$. Since $A \cap V_i \subseteq A$ and $\mu_j(A) = 0$, the set $A \cap V_i$ is also measurable w.r.t. μ_j and has measure 0: $\mu_j(A \cap V_i) = 0$.

Due to Definition 4.1, we have $\mu_i(A \cap V_i) = \lambda_{ij} \cdot \mu_j(A \cap V_i) = \lambda_{ij} \cdot 0 = 0$. So the intersection $A \cap V_i$ is a set of μ_i-measure 0, and since x is a μ_i-random element, x does not belong to $A \cap V_i$. Thus, $x \notin A$.

So x does not belong to any defineable set A of μ_j-measure 0 and therefore x is μ_j-random. QED

Comment. Proposition 2 can be proven in a similar way.

5.2. Proof of Proposition 3

In order to prove Proposition 3, let us prove the following lemma (which is in some sense inverse to Proposition 1):

Lemma. Let μ be a limit measure, $i < j$ be integers, and let $x \in V_i$ be random w.r.t. μ_j. Then x will be also random w.r.t. μ_i.

Proof of the Lemma. We need to prove that x does not belong to any definable set A of μ_i-measure 0. Indeed, if $\mu_i(A) = 0$ for some $A \subseteq V_i$, then due to Definition 4.1, we have $\mu_j(A) = \lambda_{ij}^{-1} \cdot \mu_i(A) = 0$. Since x is random

w.r.t. μ_j and the definable set A is of μ_j-measure 0, the element x cannot belong to A. The Lemma is proven.

Let us now prove the proposition itself. The proposition states that two notions of randomness are equivalent. To prove it, we will prove the implication in both directions.

• Let x be random w.r.t. μ. This means that x is random w.r.t. μ_n for some n for which $x \in V_n$. Let us show that x is random w.r.t. μ_0, i.e., that x does not belong to any definable set A of μ_0-measure 0. Indeed, let $\mu(A) = 0$. The only way for $x \in V_n$ to belong to A is to belong to $A \cap V_n$. From $\mu_0(A) = 0$ and $A \cap V_n \subseteq A$, we conclude that $\mu_0(A \cap V_n) = 0$. Since $A \cap V_n \subseteq V_n$, by definition of V_n, we have

$$\mu_n(A \cap V_n) = \lambda_{1n}^{-1} \cdot \mu_0 (A \cap V_n) = 0$$

So, $A \cap V_n$ is a set of μ_n-measure 0. Since x is μ_n-random, x cannot belong to $A \cap V_n$, and thus cannot belong to A. Therefore, x is μ_0-random.

• Let now x be random w.r.t. μ_0. Let us show that x is also random w.r.t. μ. Indeed, since $X = \cup V_i$, there exists an n such that $x \in V_n$. We will show that x is random w.r.t. μ_n and therefore random w.r.t. μ. To show that x is μ_n-random, we must show that x does not belong to any set $A \subseteq V_n$ of μ_n-measure 0. Indeed, if $\mu_n(A) = 0$ and $A \subseteq V_n$, then from the definition of μ_0, we can conclude that $\mu_0(A) = \lambda_{1n} \cdot \mu_n(A) = 0$. Since x is μ_0-random and the set A is of μ_0-measure 0, x cannot belong to A. Hence, x is μ_n-random and therefore μ-random. QED

Comment. Proposition 4 can be proven in a similar way.

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